Pertanika J. Sci. & Technol. **17 (1)**: 163 – 171 (2009)
ISSN: 0128-7680

Some Explicit Conditions for a Stationary Representation of the Unilateral Second-Order Spatial ARMA Model

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ABSTRACT

The analysis of the spatial data has been carried out in many disciplines such as demography, meteorology, geology and remote sensing. The spatial data modelling is important because it recognizes the phenomenon of spatial correlation in field experiments. Three main categories of the spatial models, namely, the simultaneous autoregressive (SAR) models (Whittle, 1954), the conditional autoregressive (CAR) models (Bartlett, 1971), and the moving average (MA) models (Haining, 1978) have been studied. Whittle (1954) presented a form of bilateral autoregressive (AR) models, whereas Basu and Reinsel (1993) considered the *first-order* autoregressive moving average (ARMA) model of the quadrant type. Awang, N. and Mahendran Shitan (2003) presented the second-order ARMA model, and established some explicit stationary conditions for the model. When fitting the spatial models and making prediction, it is assumed that, the properties of the process would not change with sites. Properties like stationarities have to be assumed, and for this reason, it was therefore imperative that the researchers had made certain that the process was stationary. This could be achieved by providing the explicit stationarity conditions for the model. The explicit conditions, for a stationary representation of the *second-order* spatial unilateral ARMA model denoted as ARMA(2,1;2,1), have been established (Awang, N. and Mahendran Shitan, 2003) and in this paper, some explicit conditions are established for a stationary representation of the *second-order* spatial unilateral ARMA model, denoted as ARMA(2,2;2,2).

Keywords: ARMA model, spatial model, stationarity

INTRODUCTION

In this study, some conditions for stationarity are established for a more general second-order autoregressive moving average model:

$$Y_{ij} = \alpha_1 Y_{i-l,j} + \alpha_2 Y_{i,j-1} + \alpha_3 Y_{i-1,j-1} + \alpha_4 Y_{i-2,j} + \alpha_5 Y_{i-2,j-1} + \alpha_6 Y_{i,j-2} + \alpha_7 Y_{i-1,j-2} + \alpha_8 Y_{i-2,j-2} + \varepsilon_{ij} + \theta_1 \varepsilon_{i-1,j} + \theta_2 \varepsilon_{i,j-1} + \theta_3 \varepsilon_{i-1,j-1}$$

$$+\theta_4 \varepsilon_{i-2,j} + \theta_5 \varepsilon_{i-2,j-1} + \theta_6 \varepsilon_{i,j-2} + \theta_7 \varepsilon_{i-1,j-2} + \theta_8 \varepsilon_{i-2,j-2}, \tag{1}$$

where Y_{ij}, the value at the site (i,j), is a finite autoregesssion of the values at the sites, which lie in the lower quadrant of (i,j), for $i = 1,\ldots, m$, $j = 1,\ldots, n$ and ε_{ij} are a collection of independent random variables, with $E(\varepsilon_{ij}) = 0$ and $\mathrm{Var}(\varepsilon_{ij}) = \sigma^2$.

Received: 11 January 2008
Accepted: 8 April 2008
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In Section 2, sufficient and necessary conditions for the existence of a stationary representation of the model in (1) are established. In Section 3, the conclusions are drawn.

THE CONDITIONS FOR THE SECOND-ORDER ARMA SPATIAL MODEL

First, the backward shift operators B_1 and B_2 were defined effectively as $B_1 Y_{ij} = Y_{i-1,j}$ and $B_2 Y_{i,j-1} = Y_{i,j-1}$. Equation (1) can then be written as follows:

$$(1- \alpha_1 B_1 - \alpha_2 B_2 - \alpha_3 B_1 B_2 - \alpha_4 B_1^2 - \alpha_5 B_1^2 B_2 - \alpha_6 B_2^2 - \alpha_7 B_1 B_2^2 - \alpha_8 B_1^2 B_2^2)\ Y_{ij}$$

$$= (1+ \theta_1 B_1 + \theta_2 B_2 + \theta_3 B_1 B_2 + \theta_4 B_1^2 + \theta_5 B_1^2\ B_2 + \theta_6 B_2^2 + \theta_7 B_1 B_2^2 + \theta_8 B_1^2 B_2^2)\ \varepsilon_{ij} \tag{2}$$

Equation (2) can be written more compactly as;

$$\Phi(B_1, B_2)\, Y_{ij} = \Theta(B_1,\ B_2)\,\varepsilon_{ij} \tag{3}$$

where

$$\Phi(B_1,B_2) = (1 - \alpha_1 B_1 - \alpha_2 B_2 - \alpha_3 B_1 B_2 - \alpha_4 B_1^2 - \alpha_5 B_1^2 B_2 - \alpha_6 B_2^2 - \alpha_7 B_1 B_2^2 - \alpha_8 B_1^2\ B_2^2)$$

and

$$\Theta(B_1,\ B_2) = (1 + \theta_1 B_1 + \theta_2 B_2 + \theta_3 B_1 B_2 + \theta_4 B_1^2 + \theta_5 B_1^2 B_2 + \theta_6 B_2^2 + \theta_7 B_1 B_2^2 + \theta_8 B_1^2\ B_2^2)$$

Proposition. For the model defined as in (1), if none of the roots of $\Phi(z_1,\ z_2) = 1 - \alpha_1 z_1 - \alpha_2 z_2 - \alpha_3 z_1 z_2 - \alpha_4 z_1^2 - \alpha_5 z_1^2\ z_2 - \alpha_6 z_2^2 - \alpha_7 z_1 z_2^2 - \alpha_8 z_1^2 z_2^2 = 0$ lie within the closed unit polydisc ($|z_1|\le 1, |z_2 \le 1$) then

(i) $|\alpha_6| - |\alpha_2| < 1$,

(ii) $|\alpha_3 + \alpha_2 + \alpha_5| > |\alpha_7 + \alpha_6 + \alpha_8 + \alpha_4 + \alpha_1 - 1|$,

(iii) $|\alpha_3 - \alpha_2 - \alpha_5| > |\alpha_7 - \alpha_6 - \alpha_8 - \alpha_4 + \alpha_1 + 1|$,

(iv) $|\alpha_4| - |\alpha_1| < 1$,

(v) $|\alpha_3 + \alpha_1 + \alpha_7| > |\alpha_5 + \alpha_4 + \alpha_8 + \alpha_6 + \alpha_2 - 1|$,

(vi) $|\alpha_3 - \alpha_1 - \alpha_7| > |\alpha_5 - \alpha_4 - \alpha_8 - \alpha_6 + \alpha_2 + 1|$,

Proof of Proposition:
(a) Sufficiency

(i) Since $\Phi(z_1,z_2) \neq 0$ for all $|z_1| \le 1, |z_2| \le 1$ it implies that for $z_1 = 0, \Phi(z_1,z_2) \neq 0$ for $|z_2| \le 1$. The roots of $\Phi(0,z_2)$ $= 1 - \alpha_2 z_2 - \alpha_6 z_2^2 = 0$ is given by $z_2 = \dfrac{-\alpha_2 \pm \sqrt{\alpha_2^2 + 4\alpha_6}}{2\alpha_6}$. However, it is required that $|z_2| > 1$. Therefore,

$$z_2 = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 + 4\alpha_6}}{2\alpha_6} > 1.$$

This will give:

$$\left|\frac{-\alpha_2 \pm \sqrt{\alpha_2^2 + 4\alpha_6}}{2\alpha_6}\right| > 1,$$

$$\left|-\alpha_2 \pm \sqrt{\alpha_2^2 + 4\alpha_6}\right| > 2|\alpha_6| \;,$$

$$\left(-\alpha_2 \pm \sqrt{\alpha_2^2 + 4\alpha_6}\right)^2 > 4\alpha_6^2 \,,$$

$$\alpha_2^2 + \alpha_2^2 + 4\alpha_6 \mp 2\alpha_2\sqrt{\alpha_2^2 + 4\alpha_6} > 4\alpha_6^2 \;,$$

$$\mp 2\alpha_2\sqrt{\alpha_2^2 + 4\alpha_6} > 4\alpha_6^2 - 2\alpha_2^2 - 4\alpha_6 \;,$$

$$4\alpha_2^2(\alpha_2^2 + 4\alpha_6) > 16\alpha_6^4 + 4\alpha_2^4 + 16\alpha_6^2 - 16\alpha_2^2\alpha_6^2 - 32\alpha_6^3 + 16\alpha_2^2\alpha_6 \,,$$

$$16\alpha_6^4 + 16\alpha_6^2 - 16\alpha_6^2\alpha_2^2 - 32\alpha_6^3 < 0,$$

$$\alpha_6^2 + 1 - \alpha_2^2 - 2\alpha_6 < 0,$$

$$(\alpha_6 - 1)^2 < \alpha_2^2,$$

$$|\alpha_6 - 1| < |\alpha_2|,$$

$$|\alpha_6| - 1 < |\alpha_2|, \text{ and finally}$$

$$|\alpha_6| - |\alpha_2| < 1.$$

This establishes condition (i).
(ii) Taking $z_1 = 1$ implies that,
$\Phi(1, z_2) = 1 - \alpha_1 - \alpha_2 z_2 - \alpha_3 z_2 - \alpha_4 - \alpha_5 z_2 - \alpha_6 z_2^2 - \alpha_7 z_2^2 - \alpha_8 z_2^2$. The roots of $\Phi(1, z_2) = 0$ are given by:

$$z_2 = \frac{-(\alpha_2 + \alpha_3 + \alpha_5) \pm \sqrt{(\alpha_2 + \alpha_3 + \alpha_5)^2 + 4(\alpha_6 + \alpha_7 + \alpha_8)(1 - \alpha_1 - \alpha_4)}}{2(\alpha_6 + \alpha_7 + \alpha_8)}$$

However it is required that $|z_2| > 1$. Hence, this will give:
$\left|-(\alpha_2 + \alpha_3 + \alpha_5) \pm \sqrt{(\alpha_2 + \alpha_3 + \alpha_5)^2 + 4(\alpha_6 + \alpha_7 + \alpha_8)(1 - \alpha_1 - \alpha_4)}\right| > 2|\alpha_6 + \alpha_7 + \alpha_8|$, which will lead to:
$z_1 = -1$, the roots of $> |\, \alpha_7 + \alpha_6 + \alpha_8 + \alpha_4 + \alpha_1 - 1|$.

This establishes condition (ii).
(iii) Taking $z_1 = -1$, the roots of $\Phi(-1, z_2) = 1 + \alpha_1 - \alpha_2 z_2 + \alpha_3 z_2 - \alpha_4 - \alpha_5 z_2 - \alpha_6 z_2^2 + \alpha_7 z_2^2 - \alpha_8 z_2^2$ $= 0$ is given by:

$$z_2 = \frac{-(\alpha_2 - \alpha_3 - \alpha_5) \pm \sqrt{(\alpha_3 - \alpha_2 - \alpha_5)^2 - 4(\alpha_7 - \alpha_6 - \alpha_8)(1 + \alpha_1 - \alpha_4)}}{2(\alpha_7 - \alpha_6 - \alpha_8)} .$$

However, it is required that $|z_2|>1$. This will give:

$\left|-(\alpha_3 - \alpha_2 - \alpha_5) \pm \sqrt{(\alpha_3 - \alpha_2 - \alpha_5)^2 - 4(\alpha_7 - \alpha_6 - \alpha_8)(1 + \alpha_1 - \alpha_4)}\right| > 2|\alpha_7 - \alpha_6 - \alpha_8|$, which leads to:

$$|\alpha_3 - \alpha_2 - \alpha_5| > |\alpha_7 - \alpha_6 - \alpha_8 - \alpha_4 + \alpha_1 + 1|.$$

This establishes condition (iii).

(iv) Taking $z_2 = 0$, the roots $z_1 = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_4}}{2\alpha_4}$ requires that $|z_1|>1$.

This produces the following:

$$4\alpha_1^2(\alpha_1^2 + 4\alpha_4) > 16\alpha_4^4 + 4\alpha_1^4 + 16\alpha_4^2 - 16\alpha_1^2\alpha_4^2 - 32\alpha_4^3 + 16\alpha_1^2\alpha_4,$$

$$16\alpha_4^4 + 16\alpha_4^2 - 16\alpha_1^2\alpha_4^2 - 32\alpha_4^3 < 0,$$

$$\alpha_4^2 + 1 - \alpha_1^2 - 2\alpha_4 < 0,$$

$$(\alpha_4 - 1)^2 < \alpha_1^2,$$

$$|\alpha_4 - 1| < |\alpha_1|, \text{ but}$$

$$|\alpha_4| - |1| < |\alpha_4 - 1|, \text{ therefore}$$

$$|\alpha_4| - 1 < |\alpha_1|,$$

$$|\alpha_4| - |\alpha_1| < 1.$$

This establishes condition (iv).

(v) Taking $z_2 = 1$ implies that:
$\Phi(z_1,1) = 1 - \alpha_1 z_1 - \alpha_2 - \alpha_3 z_1 - \alpha_4 z_1^2 - \alpha_5 z_1^2 - \alpha_6 - \alpha_7 z_1 - \alpha_8 z^2$. The roots of $\Phi(z_1,1) = 0$ are given by: $z_1 = \frac{-(\alpha_1 + \alpha_3 + \alpha_7) \pm \sqrt{(\alpha_1 + \alpha_3 + \alpha_7)^2 + 4(\alpha_4 + \alpha_5 + \alpha_8)(1 - \alpha_2 + \alpha_6)}}{2(\alpha_4 + \alpha_5 + \alpha_8)}$.

However, we require that $|z_1|>1$. This will give:

$$\left|-(\alpha_1 + \alpha_3 + \alpha_7) \pm \sqrt{(\alpha_1 + \alpha_3 + \alpha_7)^2 + 4(\alpha_4 + \alpha_5 + \alpha_8)(1 - \alpha_2 + \alpha_6)}\right| > 2|\alpha_4 + \alpha_5 + \alpha_8|,$$

which leads to:

$$|\alpha_3 + \alpha_1 + \alpha_7| > |\alpha_5 + \alpha_4 + \alpha_8 + \alpha_6 + \alpha_2 - 1|.$$

This establishes the condition (v).

(vi) Taking $z_2 = -1$ implies that:
$\Phi(z_1, -1) = 1 - \alpha_1 z_1 + \alpha_2 + \alpha_3 z_1 - \alpha_4 z_1^2 + \alpha_5 z_1^2 - \alpha_6 - \alpha_7 z_1 - \alpha_8 z_1^2$. The roots of $\Phi(z_1,-1) = 0$ are given by:

$$z_1 = \frac{-(\alpha_3 - \alpha_1 - \alpha_7) \pm \sqrt{(\alpha_3 - \alpha_1 - \alpha_7)^2 - 4(\alpha_5 - \alpha_4 - \alpha_8)(1 + \alpha_2 - \alpha_6)}}{2(\alpha_5 - \alpha_4 - \alpha_8)} .$$

However, it is required that $|z_1| > 1$. This will give:

$\left|-(\alpha_3 - \alpha_1 - \alpha_7) \pm \sqrt{(\alpha_3 - \alpha_1 - \alpha_7)^2 - 4(\alpha_5 - \alpha_4 - \alpha_8)(1 + \alpha_2 - \alpha_6)}\right| > 2|\alpha_5 - \alpha_4 - \alpha_8|$, which leads to:

$$|\alpha_3 - \alpha_1 - \alpha_7| > |\alpha_5 - \alpha_4 - \alpha_8 - \alpha_6 + \alpha_2 + 1|.$$

This establishes condition (vi).

(b) *Necessity*
In (3), if $\Phi(z_1,z_2) = 0$ has root (w_1, w_2), then:

$$(1 - \alpha_1 w_1 - \alpha_2 w_2 - \alpha_3 w_1 w_2 - \alpha_4 w_1^2 - \alpha_5 w_1^2 w_2 - \alpha_6 w_2^2 - \alpha_7 w_1 w_2^2 - \alpha_8 w_1^2 w_2^2) = 0.$$

This gives

$$w_2 = \frac{-(\alpha_2 + \alpha_3 w_1 + \alpha_5 w_1^2) \pm \sqrt{(\alpha_2 + \alpha_3 w_1 + \alpha_5 w_1^2)^2 - 4(\alpha_6 + \alpha_7 w_1 + \alpha_8 w_1^2)(\alpha_1 w_1 + \alpha_4 w_1^2 - 1)}}{2(\alpha_6 + \alpha_7 w_1 + \alpha_8 w_1^2)} \tag{4}$$

Equation(4) can also be rewritten as, $w_2 = \dfrac{-b_2 \pm \sqrt{b_2^2 - 4a_2 c_2}}{2a_2}$, where:
$a_2 = \alpha_6 + \alpha_7 \operatorname{Re}(w_1) + \alpha_8 |w_1|^2$, $b_2 = \alpha_2 + \alpha_3 \operatorname{Re}(w_1) + \alpha_5 |w_1|^2$ and $c_2 = \alpha_1 \operatorname{Re}(w_1) + \alpha_4 |w_1|^2 - 1$.

Let $|w_1|$, and write $w_1 = r\exp(i\phi)$, where $0 \le r \le 1$, $0 \le \phi \le 2\pi$.
We than has:
$|r\exp(i\phi)| < 1$ or $|r| < 1$. This is equivalent to $-1 < r < 1$.
The researchers need to establish that no roots of $\Phi(z_1, z_2) = 0$ lie within the closed unit polydisc, and hence, it we need to show that:

$|w_2| > 1$ or $|w_2|^2 > 1$.

For $|w_2|^2 > 1$, we have $\left|\dfrac{-b_2 \pm \sqrt{b_2^2 - 4a_2 c_2}}{2a_2}\right|^2 > 1$, or $b_2^2 > (a_2 + c_2)^2$. This means the following is needed:

$$(\alpha_2 + \alpha_3 \operatorname{Re}(w_1) + \alpha_5 |w_1|^2)^2 > (\alpha_6 + \alpha_7 \operatorname{Re}(w_1) + \alpha_8 |w_1|^2 + \alpha_1 \operatorname{Re}(w_1) + \alpha_4 |w_1|^2 - 1)^2$$

or

$$\alpha_2^2 + \alpha_3^2 r^2 + \alpha_5^2 r^4 + 2(|\alpha_2\alpha_3| r + |\alpha_2\alpha_5| r^2 + |\alpha_3\alpha_5| r^3) > \alpha_6^2 + \alpha_7^2 r^2 + \alpha_8^2 r^4$$
$$+ 2(|\alpha_6\alpha_7| r + |\alpha_7\alpha_8| r^3 + |\alpha_6\alpha_8| r^2)$$

$$+\alpha_1^2 r^2 + \alpha_4^2 r^4 + 1 + 2\,(|\alpha_1\alpha_4|r^3 - |\alpha_1|r - |\alpha_4|r^2)$$
$$+ 2(|\alpha_1\alpha_6|r + |\alpha_4\alpha_6|r^2 - |\alpha_6| + |\alpha_1\alpha_7|r^2 + |\alpha_4\alpha_7|r^3 - |\alpha_7|r + |\alpha_1\alpha_8|r^3 + |\alpha_4\alpha_8|r^4 - |\alpha_8|r^2)$$

Nevertheless, it is sufficient to show that:

$$\alpha_2^2 - \alpha_6^2 - 1 + 2\,|\alpha_6| > M,$$

where $M = \sup_{0\le r\le 1} f(r)$

$$= \sup_{0\le r\le 1} \{\alpha_7^2 r^2 + \alpha_8^2 r^4 + 2(|\alpha_6\alpha_7|r + |\alpha_7\alpha_8|r^3 + |\alpha_6\alpha_8|r^2)$$
$$+ \alpha_1^2 r^2 + \alpha_4^2 r^4 + 2(|\alpha_1\alpha_4|r^3 - |\alpha_1|r - |\alpha_4|r^2)$$
$$+ 2(|\alpha_1\alpha_6|r + |\alpha_4\alpha_6|r^2 + |\alpha_1\alpha_7|r^2 + |\alpha_4\alpha_7|r^3 - |\alpha_7|r + |\alpha_1\alpha_8|r^3 + |\alpha_4\alpha_8|\alpha_7|r^4 - |\alpha_8|r^2)$$
$$- \alpha_3^2 r^2 - \alpha_5^2 r^4 - 2(|\alpha_2\alpha_3|r + |\alpha_2\alpha_5|r^2 + |\alpha_3\alpha_5|r^3)\}$$

The function $f(r)$ may attain its minimum over, $0 \le r \le 1$ at $r = 0$, in which case $\alpha_2^2 - \alpha_6^2 - 1 + 2\,|\alpha_6| > 0$, or $|\alpha_6| - |\alpha_2| < 1$, is needed, and this follows condition (i):
Similarly, the function $f(r)$ may attain its maximum over $0 \le r \le 1$ at $r = 1$, in which, the following is needed:

$$\alpha_2^2 - \alpha_6^2 - 1 + 2|\alpha_6| > \alpha_7^2 + \alpha_8^2 + 2\,(|\alpha_6\alpha_7| + |\alpha_7\alpha_8| + |\alpha_6\alpha_8|)$$
$$+ \alpha_1^2 + \alpha_4^2 + 2(|\alpha_1\alpha_4| - |\alpha_1| - |\alpha_4|)$$
$$+ 2(|\alpha_1\alpha_6| + |\alpha_4\alpha_6| + |\alpha_1\alpha_7| + |\alpha_4\alpha_7| - |\alpha_7| + |\alpha_1\alpha_8| + |\alpha_4\alpha_8| - |\alpha_8|) - \alpha_3^2 - \alpha_5^2$$
$$- 2\,(|\alpha_2\alpha_3| + |\alpha_2\alpha_5| + |\alpha_3\alpha_5|).$$

This can also be re-expressed as:

$$\alpha_2^2 + \alpha_3^2 + \alpha_5^2 + 2(|\alpha_2\alpha_3| + |\alpha_2\alpha_5| + |\alpha_3\alpha_5| > \alpha_6^2 + \alpha_7^2 + \alpha_8^2 + \alpha_1^2 + \alpha_4^2 + 1$$
$$+ 2(|\alpha_6\alpha_7| + |\alpha_7\alpha_8| + |\alpha_4\alpha_7| + |\alpha_1\alpha_7| - |\alpha_7|)$$
$$+ 2(|\alpha_6\alpha_8| + |\alpha_4\alpha_6| + |\alpha_1\alpha_6| - |\alpha_6|)$$
$$+ 2\,(|\alpha_4\alpha_8| + |\alpha_1\alpha_8| - |\alpha_8|) + 2|\alpha_1\alpha_4| - 2|\alpha_4| - 2|\alpha_1|).$$

From the above $|\alpha_3 + \alpha_2 + \alpha_5| > |\alpha_7 + \alpha_6 + \alpha_8 + \alpha_4 + \alpha_1 - 1|$, can be obtained, and this follows condition (ii).

The function $f(r)$ may also attain its maximum over $0 \le r \le 1$ at $r = 1$, in which the following is needed:

$$\alpha_2^2 - \alpha_6^2 - 1 - 2|\alpha_6| > \alpha_7^2 + \alpha_8^2 + 2\,(-|\alpha_6\alpha_7| - |\alpha_7\alpha_8| + |\alpha_6\alpha_8|)$$
$$+ \alpha_1^2 + \alpha_4^2 + 2(-|\alpha_1\alpha_4| + |\alpha_1| - |\alpha_4|)$$
$$+ 2(-|\alpha_1\alpha_6| + |\alpha_4\alpha_6| + |\alpha_1\alpha_7| - |\alpha_4\alpha_7| + |\alpha_7| - |\alpha_1\alpha_8| + |\alpha_4\alpha_8| - |\alpha_8|) - \alpha_3^2 - \alpha_5^2$$
$$- 2\,(-|\alpha_2\alpha_3| + |\alpha_2\alpha_5| - |\alpha_3\alpha_5|).$$

This can also be re-expressed as:

$$\alpha_2^2 + \alpha_3^2 + \alpha_5^2 - 2(|\alpha_2\alpha_3| - |\alpha_2\alpha_5| + |\alpha_3\alpha_5| > \alpha_7^2 + \alpha_6^2 + \alpha_8^2 + \alpha_1^2 + \alpha_4^2 + 1$$
$$+ 2(-|\alpha_6\alpha_7| - |\alpha_7\alpha_8| + |\alpha_1\alpha_7| - |\alpha_4\alpha_7| + |\alpha_7|)$$
$$+ 2(|\alpha_6\alpha_8| - |\alpha_1\alpha_6| + |\alpha_4\alpha_6| - |\alpha_6|)$$
$$+ 2\,(-|\alpha_1\alpha_8| + |\alpha_4\alpha_8| - |\alpha_8|) + 2(-|\alpha_1\alpha_4| + |\alpha_1|) - 2|\alpha_4|.$$

From this $|\alpha_3 - \alpha_2 - \alpha_5| > |\alpha_7 - \alpha_6 - \alpha_8 + \alpha_1 - \alpha_4 + 1|$, can be obtained and this follows condition (iii).

To provide proofs of necessity for conditions (iv) – (vi), (3) will be considered if $\Phi(z_1, z_2) = 0$ has roots (w_1, w_2), then

$$(1 - \alpha_1 w_1 - \alpha_2 w_2 - \alpha_3 w_1 w_2 - \alpha_4 w_1^2 - \alpha_5 w_1^2 w_2 - \alpha_6 w_2^2 - \alpha_7 w_1 w_2^2 - \alpha_8 w_1^2 w_2^2) = 0.$$

This gives

$$w_1 = \frac{-(\alpha_1 + \alpha_3 w_2 + \alpha_7 w_2^2) \pm \sqrt{(\alpha_1 + \alpha_3 w_2 + \alpha_7 w_2^2)^2 - 4(\alpha_4 + \alpha_5 w_2 + \alpha_8 w_2^2)(\alpha_2 w_2 + \alpha_6 w_2^2 - 1)}}{2(\alpha_4 + \alpha_5 w_2 + \alpha_8 w_2^2)} \tag{5}$$

Equation (5) can also be rewritten as $w_1 = \dfrac{-b_1 \pm \sqrt{b_1^2 - 4a_1 c_1}}{2a_1}$, where:

$a_1 = \alpha_4 + \alpha_5 \operatorname{Re}(w_2) + \alpha_8 |w_2|^2$, $b_1 = \alpha_1 + \alpha_3 \operatorname{Re}(w_2) + \alpha_7 |w_2|^2$ and $c_1 = \alpha_2 \operatorname{Re}(w_2) + \alpha_6 |w_2|^2 - 1$.

Let $|w_2| \le 1$, and write $w_2 = r\exp(i\phi)$, where $0 \le r \le 1$, $0 \le \phi \le 2\pi$.

Since $|r\exp(i\phi)| < 1$ or $|r| < 1$, This is equivalent to $-1 < r < 1$.

Similarly, it is crucial to establish that no roots of $\Phi(z_1, z_2) = 0$ lie within the closed unit polydisc, and for this reason, it is therefore important to show that:

$|w_1| > 1$ or $|w_1|^2 > 1$.

For $|w_1|^2 > 1$, $\left|\dfrac{-b_1 \pm \sqrt{b_1^2 - 4a_1 c_1}}{2a_1}\right|^2 > 1$, or $b_1^2 > (a_1 + c_1)^2$ is achieved.

This means, the following is required:

$$(\alpha_1 + \alpha_3 \operatorname{Re}(w_2) + \alpha_7 |w_2|^2)^2 > (\alpha_4 + \alpha_5 \operatorname{Re}(w_2) + \alpha_8 |w_2|^2 + \alpha_2 \operatorname{Re}(w_2) + \alpha_6 |w_2|^2 - 1)^2$$

or

$$\begin{aligned}
&\alpha_1^2 + \alpha_3^2 r^2 + \alpha_7^2 r^4 + 2(|\alpha_1 \alpha_3| r + |\alpha_1 \alpha_7| r^2 + |\alpha_3 \alpha_7| r^3) > \\
&\alpha_4^2 + \alpha_5^2 r^2 + \alpha_8^2 r^4 + \alpha_2^2 r^2 + \alpha_6^2 r^4 + 1 \\
&+ 2(|\alpha_4 \alpha_5| r + |\alpha_4 \alpha_8| r^2 + |\alpha_2 \alpha_4| r + |\alpha_4 \alpha_6| r^2 - |\alpha_4|) \\
&+ 2(|\alpha_5 \alpha_8| r^3 + |\alpha_2 \alpha_5| r^2 + |\alpha_5 \alpha_6| r^3 - |\alpha_5| r) \\
&+ 2\ (|\alpha_2 \alpha_8| r^3 + |\alpha_6 \alpha_8| r^4 - |\alpha_8| r^2) \\
&+ 2\ (|\alpha_2 \alpha_6| r^3 - |\alpha_2| r) - 2|\alpha_6| r^2
\end{aligned}$$

It is sufficient that to show that:

$$\begin{aligned}
&\alpha_1^2 - \alpha_4^2 + 2|\alpha_4| - 1 > M, \text{ where } M = \sup_{0 \le r \le 1} f(r) = \sup_{0 \le r \le 1} \{\alpha_5^2 r^2 + \alpha_8^2 r^4 + \alpha_2^2 r^2 + \alpha_6^2 r^4 \\
&+ 2(|\alpha_4 \alpha_5| r + |\alpha_4 \alpha_8| r^2 + |\alpha_2 \alpha_4| r + |\alpha_4 \alpha_6| r^2) \\
&+ 2(|\alpha_5 \alpha_8| r^3 + |\alpha_2 \alpha_5| r^2 + |\alpha_5 \alpha_6| r^3 - |\alpha_5| r) \\
&+ 2(|\alpha_2 \alpha_8| r^3 + |\alpha_6 \alpha_8| r^4 - |\alpha_8| r^2) \\
&+ 2(|\alpha_2 \alpha_6| r^3 - |\alpha_2| r) - 2\alpha_6 r^2 - \alpha_3^2 r^2 - \alpha_7^2 r^4 \\
&- 2(|\alpha_1 \alpha_3| r + |\alpha_1 \alpha_7| r^2 + |\alpha_3 \alpha_7| r^3)\}
\end{aligned}$$

The function $f(r)$ may attain its minimum over $0 \le r \le 1$ at $r = 0$, in which $\alpha_1^2 - \alpha_4^2 + 2|\alpha_4| - 1 > 0$ or $|\alpha_4| - |\alpha_1| < 1$ is needed, and this follows condition (iv).

In addition, the function $f(r)$ may attain its maximum over $0 \le r \le 1$, in which the following is needed:

$$\begin{aligned}
&\alpha_1^2 - \alpha_4^2 + 2|\alpha_4| - 1 > \alpha_5^2 + \alpha_8^2 + \alpha_2^2 + \alpha_6^2 \\
&+ 2(|\alpha_4\alpha_5| + |\alpha_4\alpha_8| + |\alpha_2\alpha_4| + |\alpha_4\alpha_6|) \\
&+ 2(|\alpha_5\alpha_8| + |\alpha_2\alpha_5| + |\alpha_5\alpha_6| - |\alpha_5|) \\
&+ 2(|\alpha_2\alpha_8| + |\alpha_6\alpha_8| - |\alpha_8|) \\
&+ 2(|\alpha_2\alpha_6| - |\alpha_2|) - 2\alpha_6 + \alpha_3^2 - \alpha_7^2 \\
&+ 2(|\alpha_1\alpha_3| + |\alpha_1\alpha_7| + |\alpha_3\alpha_7|)
\end{aligned}$$

This can also be re-expressed as:

$$\begin{aligned}
&\alpha_1^2 + \alpha_3^2 + \alpha_7^2 + 2(|\alpha_1\alpha_3| + |\alpha_1\alpha_7| + |\alpha_3\alpha_7| > \alpha_4^2 + \alpha_5^2 + \alpha_8^2 + \alpha_2^2 + \alpha_6^2 + 1 \\
&+ 2(|\alpha_4\alpha_5| + |\alpha_4\alpha_8| + |\alpha_2\alpha_4| + |\alpha_4\alpha_6| - |\alpha_4|) \\
&+ 2(|\alpha_5\alpha_8| + |\alpha_2\alpha_5| + |\alpha_5\alpha_6| - |\alpha_5|) \\
&+ 2(|\alpha_2\alpha_8| + |\alpha_6\alpha_8| - |\alpha_8|) + 2\ (|\alpha_2\alpha_6| - 2|\alpha_2| - 2|\alpha_6|).
\end{aligned}$$

From this $|\alpha_1 + \alpha_3 + \alpha_7| > |\alpha_4 + \alpha_5 + \alpha_8 + \alpha_2 + \alpha_6 - 1|$, can be obtained, and this is based on condition (v).

The function $f(r)$ may also attain its maximum over $0 \le r \le 1$ at $r = -1$, in which the following is needed:

$$\begin{aligned}
&\alpha_1^2 - \alpha_4^2 + 2|\alpha_4| - 1 > \alpha_5^2 + \alpha_8^2 + \alpha_2^2 + \alpha_6^2 \\
&+ 2(-|\alpha_4\alpha_5| + |\alpha_4\alpha_8| - |\alpha_2\alpha_4| + |\alpha_4\alpha_6|) \\
&+ 2(-|\alpha_5\alpha_8| + |\alpha_2\alpha_5| - |\alpha_5\alpha_6| + |\alpha_5|) \\
&+ 2(-|\alpha_2\alpha_8| + |\alpha_6\alpha_8| - |\alpha_8|) \\
&+ 2(-|\alpha_2\alpha_6| + |\alpha_2|) - 2|\alpha_6| - \alpha_3^2 - \alpha_7^2 \\
&- 2(-|\alpha_1\alpha_3| + |\alpha_1\alpha_7| - |\alpha_3\alpha_7|)
\end{aligned}$$

This can be re-expressed as:

$$\begin{aligned}
&\alpha_1^2 + \alpha_3^2 + \alpha_7^2 + 2(-|\alpha_1\alpha_3| + |\alpha_1\alpha_7| - |\alpha_3\alpha_7|) > \alpha_4^2 + \alpha_5^2 + \alpha_8^2 + \alpha_2^2 + \alpha_6^2 + 1 \\
&+ 2(-|\alpha_4\alpha_5| - |\alpha_5\alpha_8| - |\alpha_5\alpha_6| + |\alpha_2\alpha_5| + |\alpha_5|) \\
&+ 2(|\alpha_4\alpha_8| + |\alpha_4\alpha_6| - |\alpha_2\alpha_4| - |\alpha_4|) \\
&+ 2(|\alpha_6\alpha_8| - |\alpha_2\alpha_8| - |\alpha_8|) + 2\ (-|\alpha_2\alpha_6| - |\alpha_6|) + 2|\alpha_2|).
\end{aligned}$$

From the above, $|\alpha_3 - \alpha_1 - \alpha_7| > |\alpha_5 - \alpha_4 - \alpha_8 - \alpha_6 + \alpha_2 + 1|$ can be obtained; this follows condition (vi).

CONCLUSIONS

In this study, some explicit conditions were established for the existence of a stationary representation of the more general second-order unilateral spatial ARMA model, as discussed in Section 2.

The previous models being studied were mostly of the first-order, that is, only the nearest neighbouring sites are used to model the value of a particular site. In some situations, it is not enough to merely use just the neighbouring values to model the value of a certain site or describe the spatial correlations in the data. Therefore, not only a second-order spatial model is necessary, it can also serve as an alternative to depict the spatial correlation of the data on a regular grid. However, the estimation of the parameters of such a model must be done in such a way that the conditions (set out in Section 2) do not contradict and that the parameters of the model should be maximised with bounded constraints.

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